Graph Theory, Class 1

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1 Definitions and Notations

Definition 1.1 (Degree)**.** Given a vertex v, the **degree** of v is defined to be the number of edges containing v as an endpoint.

Definition 1.2 (Complement Graph)**.** Let $G = (V, E)$ be a graph. The **complement** $\bar{G}$ of G is a graph with the same vertex set as G and $E(\bar{G}) = \{e \notin E(G)\}$. i.e. $\bar{G}$ has edges exactly where there are no edges in G.

Definition 1.3. Let G be a connected graph. Then G contains a subgraph which is a tree containing all the vertices of G. Such a subgraph is called **spanning tree** of G.

- Sometimes its easier to solve some problem if you consider some spanning tree of the given graph because trees are easier to deal with.
- Some of the most occurred spanning trees are dfs and bfs trees (both have theirs own perks).

Definition 1.4 (Tournament Graphs)**.** A directed graph $G(V, E)$ is called tournament *iff* for every pair $(a, b) \mid 1 \leqslant a < b \leqslant |V|$, there exists either a directed edge from a to b or a directed edge from b to a. In other words, A tournament is defined to be a complete directed graph.

Definition 1.5 (Hamiltonian Path)**.** A path that visits every vertex **exactly** once.

Definition 1.6 (Hamiltonian Cycle)**.** A Cycle that visits every vertex **exactly** once.

2 Notations

1. Maximum degree of any node in graph G is denoted with $\Delta(G)$.

3 Problems

3.1 Basic Stuff

Problem 3.1. Let G be a graph with n vertices, m edges and the degrees of the n vertices are $d_1, d_2, \cdots, d_n$ Prove that

$$\sum_{i=1}^{n} d_i = 2m.$$

Problem 3.2. Let G be a disconnected graph. Prove that its complement $\bar{G}$ is connected.

Problem 3.3. Prove that a graph is bipartite if and only if it does not contain an odd cycle.

Problem 3.4. Characterize graphs with n vertices and n edges (connected or disconnected).

Problem 3.5. Characterize graphs with $\Delta(G) \leqslant 2$.

3.2 Induction

Problem 3.6. Given an undirected connected graph $G(V, E)$, find a subset S of E, such that in the graph $G'(V, S)$, every node has odd degree. Also, find a condition necessary and sufficient condition for this subset to exist.

Problem 3.7. Prove that every tournament graph has a Hamiltonian path.

Problem 3.8 (Sergey's Problem)**.** Prove that, for every directed graph $G(V, E)$ without self-loop, there exists $Q \subseteq V$, such that there is no directed edge between two vertices from Q and every vertex not belonging to Q can be reached from a vertex $\in Q$ in at most two moves.

3.3 Coloring

Problem 3.9. For a $n \times m$ grid, find necessary and sufficient condition for is to have a Hamiltonian Cycle. (you can move from one cell to another *iff* they share a side)